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# Modeling and Simulation of Subbase Improvements to Existing Clay for Rigid Pavement

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Challenge

Customer has an over-budget situation for paving a laydown yard due to an unanticipated specification. Customer needs to create the equivalent stiffness of the current specification using a more economical approach. This should be accomplished with minimal changes to standard operating procedures for all involved if possible.

Current Specification
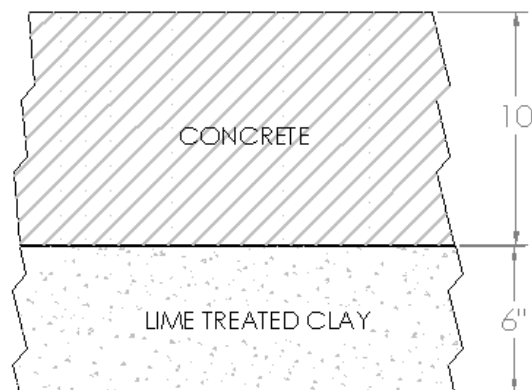
Figure 1. Current specified method

Concrete is to be 3,500 psi concrete (at 28 days). Rebar is specified as #5 rebar at 24" OCBW centered in the concrete thickness. This concrete is specified to be placed on a subbase scarified to 6", stabilized with 36 pounds per square yard of lime and compacted to 95% maximum dry density. Below the treated subbase there is 10' of FAT clay and some FAT clay with sand.

The slab will be exposed to fully loaded COMBiLift, C26000 type, forklifts. These lifts do 60 passes per day per design lane. The actual tire pressure loading can be seen in the Figure 2 below.
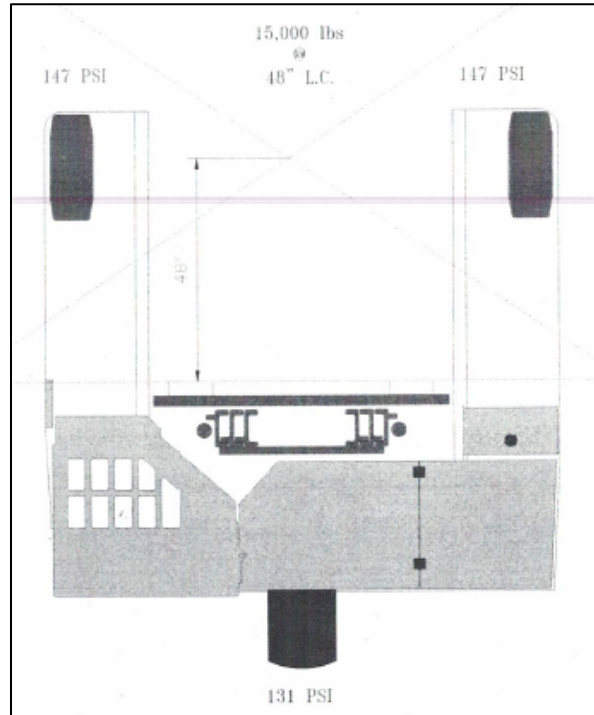
Figure 2. COMBiLift tire pressure schematic

Solution

This paper analyzes three proposed methods of using base and subbase improvements and/or a higher than specified compressive strength concrete mix to achieve the same "composite" stiffness as current specification. This paper proposes incorporating corrosion resistant basalt fiber reinforced polymer rebar (BFRP – GatorBar by Neuvokas Corp) to improve the concrete crack development and virtually eliminate the life shortening effects of steel rebar corrosion.   These combinations of material properties were used to form a multilayered composite, slab-on-grade structure that could offer promising and economical alternatives to the proposed 10" concrete slab on 6" of lime-treated base. Each of the proposed methods can be seen together in Figure 3.

Proposed Method 1   *Proposed Method 2*   *Proposed Method 3*
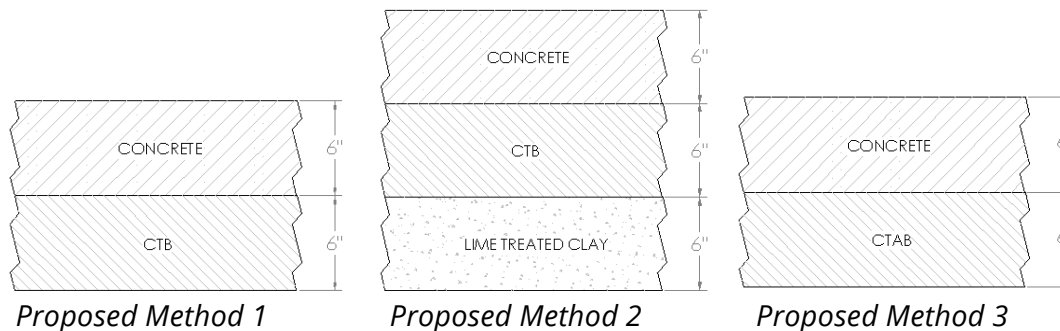
Figure 3. Proposed methods presented in this paper

This paper will also analyze the deflection and maximum stress assuming the COMBiLift loading that is described above in the current specification.

Neuvokas has prepared a multiphysics model with Michigan Technological University (MTU) that analyzes the current specification compared to the proposals listed below. This model uses inputs such as subbase, base, and concrete stiffness and loading from the COMBilift that will be used in this steel laydown yard. The model then outputs a "composite" stiffness, slab deflection, and slab ultimate stress. The "composite" stiffness can be understood as the completed stiffness of the support system from concrete slab to subbase materials.

A budgetary cost of each potential solution is also analyzed.

Proposal 1

1) 6" of 5,000 psi concrete reinforced with #3 GatorBar placed at 1" above the center of the slab thickness at 18" OCBW.   Contraction saw cuts should be 1.5 inches deep. This concrete will result in an almost 20% improvement in concrete elastic modulus and modulus of rupture

2) 6" of cement treated (CTB) base. See Table 1 for material properties.

It is understood that to maximize the effect of cement in a soil it is necessary to have greater than 50% of the soil retained over 200 mesh (75 micron). Cleaned sand would meet this standard and potentially provide an economical aggregate when mixed at a 50/50 ratio with the site clay soils (Little, 2009). In this approach Neuvokas proposed preparing the base through removing 3" of soil below the subbase level, tilling 3" below the cut line and then adding 3" of cleaned sand (>200 mesh). 6% by weight of Portland cement would be added. This mixture would then be thoroughly mixed and wetted, after which it would be pulverized and compacted in place.

Proposal 2

1) 6" of 5,000 psi concrete reinforced with #3 GatorBar placed at 1" above the center of the slab thickness at 18" OCBW

2) 6" of cement treated (CTB) base. See Table 1 for material properties.

3) 6" of lime stabilized subgrade

In this proposal Neuvokas looked at a three layer composite created with a 6″ subbase treated with lime as specified in the current plans covered with a 6″ base of imported CTB (made from site spoils and imported sand as in proposal one) after which it would be covered with a 6″ top slab. The resulting composite slab would be 18″ thick.

Proposal 3

1) 6″ of 5,000 psi concrete reinforced with #3 GatorBar placed at 1″ above the center of the slab thickness at 18″ OCBW.

2) 6″ of cement treated aggregate base (CTAB). See Table 1 for material properties.

This proposal requires creating a six-inch CTAB base. CTAB is defined as a mixture of aggregate material and a measured amount of Portland cement with water that hardens after compaction. This method is commonly used in flexible or rigid roadway pavements. The amount of cement and coarse aggregates will directly relate to the final strength and stiffness of the subbase. CTAB can show elastic, slab-like response to loading.

Assumptions

	Strength psi	Young Modulus ksi	Modulus of Rupture psi	Modulus of Subgrade Reaction psi/in
Concrete 1	3,500	3,409	444	
Concrete 2	5,000	4,074	530	
Untreated Clay Soil		1.4		low
Lime-Treated Clay Soil		75.0		>350
Cement-Treated Clay/Sand Soil		224.8		>350
Cement-Treated Aggregate Base		1,450.0		>350

Table 1. Assumptions used for materials in this study

Material properties were determined from available field data and also from values documented in the literature when field data are inaccessible. The moduli of elasticity for cement-treated and lime treated clays were assumed to be 1.55 GPa and 517 MPa, respectively (Bhattacharja and Bhatty, 2003; Tuleubekov and Brill). The corresponding untreated soil is a Texas soil which has a Young's modulus of 10 MPa, a liquid limit of 62 and a plastic limit of 22 (Bhattacharja and Bhatty, 2003; GeotechData.info). The clay subgrade has a thickness of 10 feet.

Slab loading is by using the Combi-lift C30,000 ground pressure. This machine when loaded with 15,000 lbs (total GVW of 30,000) results in 147psi at dual front tires and 133 psi at rear single tire, see image in Current Specification section of this document.

Soil Mechanics Analysis
The deflection on the pavement surface describes the degree to which the pavement structure is displaced under a load. Using one specific level of loading, a greater deflection indicates a lower stiffness of the system which consists of pavement, base, and subgrade layers. To characterize the stiffness of the system, a "composite" stiffness is defined in a way similar to that of the modulus of subgrade reaction, for which a plate is placed on the pavement for loading. In this case, we replaced the plate with a ground pressure of 147 psi, which is caused by the COMBiLift vehicle. The "composite" stiffness is calculated as the ratio between the pressure and the maximum deflection measured on the pavement surface. For simplicity, a circular loaded area with a radius of 0.1 m is chosen to approximate both the ground pressure cause by a normal vehicle tire and the loading used for modulus of subgrade reaction and for the falling weight deflectometer tests.

Thus the deflection of the pavement structure and underlying subgrade is a very typical multi-region linear elastic problem. For this elastic material, the general constitutive relationship is Hooke's law:

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor, and $\mathbf{C}$ is the fourth-order stiffness tensor. However, in engineering applications, we usually do not deal with the above equation using the stress and strain as second order tensors and the stiffness as the forth order tensors. Here we assume both concrete and soils including treated one are homogeneous and isotropic materials, then the stiffness tensor can be written using the Vigot notation as:

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}
=
\begin{bmatrix}
2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\
\lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu
\end{bmatrix}
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}
$$

where μ and λ are Lamé constants. These two constants can be formulated as functions of Young's modulus and Poisson's ratio, which we are more familiar with.

In index notation, we can write the above equation as

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}.$$

The governing equation for the mechanical field can obtained on the balance law and is also called Navier-Cauchy equations or the elastostatic equations. The above equation is also frequently formulated using the tensor notation as

$$\mu \nabla^2 \mathbf{u} + (\mu + \lambda) \nabla (\nabla \cdot \mathbf{u}) + \mathbf{F} = 0.$$

This is the primary equation used for the simulation.

Three-dimensional finite element analysis was conducted for a computational domain of 100 m by 100 m by 5 m. Such a large computational domain was first selected to ensure that the area to be analyzed is large enough to eliminate the boundary effect. The trial simulation results indicated that deformation caused by the aforementioned load only obviously impacts an area that is 20 to 30 m away from the center of the load, see Figure 4. Then the above 3D simulation is reduced into equivalent 2D simulation based on axial-symmetry of the problem.
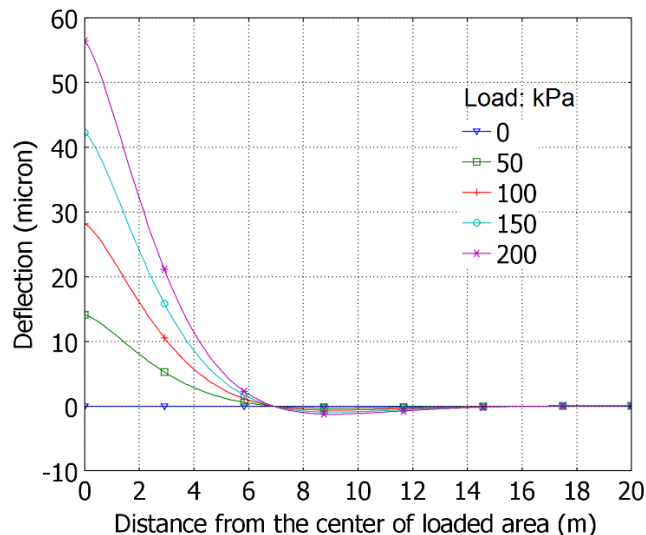
Figure 4. Surface deflection of concrete under varying loads

Figure 5 shows the stress and deflection when the concrete slab under load. The deflection in this Figure has been multiplied by 10,000 times to aid in the visualization.
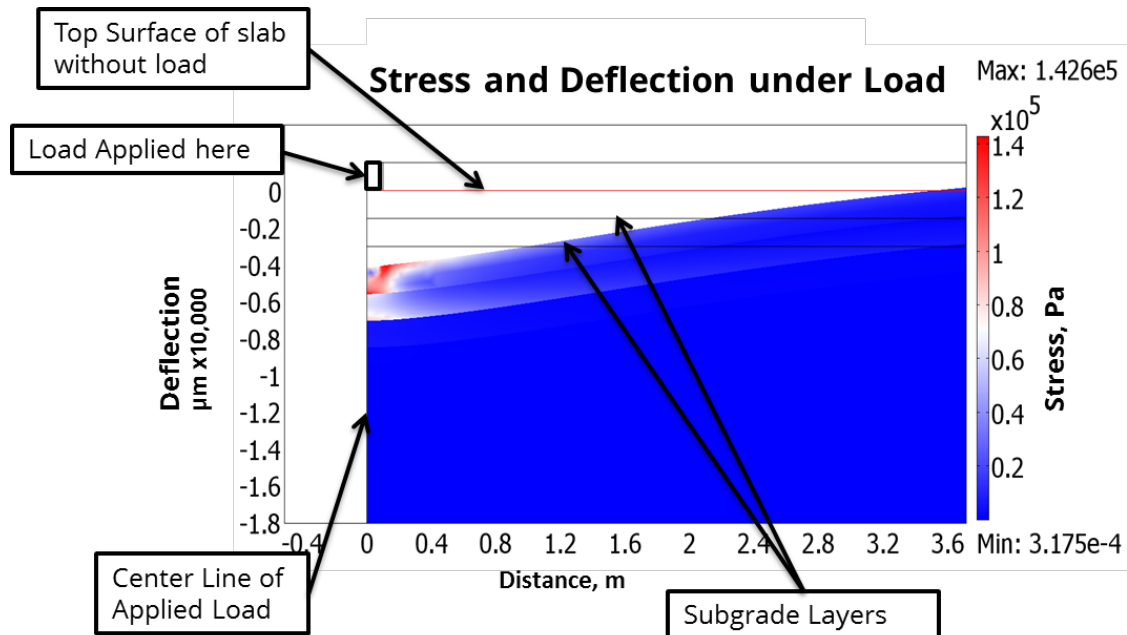
Figure 5. Stress within the pavement structure, displacement magnified by 10,000 times.

	Base/Subbase Treatement	Cement Content in Base/Subbase	Composite Stiffness kPa/m	Slab Deflection inch	Slab Maximum Stress psi
Current Specification	lime	n/a	318.8	0.013	123.1
Proposal 1, 9% Cement	cement	9%	180.6	0.022	298.3
Proposal 2, 9% cement	cement/lime	9%	205.6	0.018	241.1
Proposal 3a, 3500 psi concrete	CTAB	6%	313.7	0.013	113.5
Proposal 3b, 5000 psi concrete	CTAB	6%	325.2	0.012	105.9

Table 2. Results of soil mechanics analysis

Soil Mechanics Analysis Discussion

Proposal 1 and 2
Analysis shows that the thickness of the pavement layer is the dominant factor for determining the "composite" stiffness. Proposal 1 and Proposal 2 are modeled with six inches of concrete vs. ten inches of concrete and the resulting "composite" stiffness values where lower than that of the current specification. The thickness and stiffness of the lime-stabilized clay layer had minimal effect on the "composite" stiffness because the stiffness of the lime-stabilized clay is less than 100 psi, Table 1, and this is many times less than the stiffness of the concrete at 4,074 psi for 5,000 psi concrete. Even using a cement-treated clay subbase it will be difficult to match the "composite" stiffness that 10 inches of concrete can offer. This is based on the literature searching of values that offer up to 225 psi, Table 1, elastic modulus.

It should be noted that even though the "composite" stiffness is much less than the current specification that the deflection and slab maximum stress is still relatively low. The maximum stress values are lower than the modulus of rupture, 530 psi, for 5000 psi compressive strength concrete. As can be seen in Table 3 this creates a safety factor of 1.8 for Proposal 1 and a safety factor of 2.2 for Proposal 2. This means that the slab will not fail under this loading. When considering slab deformations ride quality is often an important factor. When considering the type and quantity of traffic this slab would see, the deformations seen in Table 2 should not be an issue.

	Base/Subbase Treatement	Slab Maximum Stress psi	Slab Modulus of Rupture psi	Safety Factor
Current Specification, 3500 psi concrete	lime	123.1	444	3.6
Proposal 1, 9% Cement	cement	298.3	530	1.8
Proposal 2, 9% cement	cement/lime	241.1	530	2.2
Proposal 3a, 3500 psi concrete	CTAB	105.9	444	4.2
Proposal 3b, 5000 psi concrete	CTAB	113.5	530	4.7

Table 3. Safety Factor when considering slab maximum stress and modulus of rupture

Proposal 3
Whether using 3500 psi or 5000 psi compressive strength concrete, Proposal 3 offers an alternative that can closely match the "composite" stiffness of the current specification with 10 inches of concrete. Using a CTAB modulus of 1,450 psi the desired stiffness can be achieved (Lim and Zollinger, 2003). This is a significant increase in stiffness that cannot be achieved by simply treating the existing clay. As Table 3 shows that the safety factor, when considering modulus of rupture for the 3,500 psi and 5,000 psi concrete, will exceed the current specification.

Neuvokas believes that some amount of aggregate (crushed limestone) could be added to the existing clay, with cement then added as well, to create a base somewhere between CTB and a complete CTAB. Testing would be needed to confirm the elastic modulus of the final mix design for this base.

This type of base will increase the coefficient of friction between the concrete and base material. Typically this can create more cracking in the slab-on-grade (Chen, 2003), but using Neuvokas GatorBar will reduce the amount of cracking that could occur from a base material such as CTAB.

Rebar

Neuvokas GatorBar is an alternative to black steel rebar in a many applications. It offers the performance advantages such as zero rust and lower weight. Table 4 shows one advantage of this weight savings, #5 steel rebar only offers 1,912 feet of rebar per ton where #3 GatorBar can offer 21,858 feet of rebar per ton.

	Rebar Size	Area of Rebar in^2	Quantity of rebar per ton feet	Tensile Strength psi	Cost of rebar per foot
Steel rebar	#5	0.307	1,912	60,000	$0.38
Gatorbar	#3	0.11	21,858	145,000	$0.25
Assuming #5 steel rebar at $720/ton					

Table 4. Steel rebar vs. GatorBar product information.

As can be seen in Table 5, calculations show that #5 black steel rebar (60 ksi tensile) on 24" centers provides 18,420 lbs of tensile restraint to the slab at a reinforcement ratio of .51%. The same calculation shows that #3 GatorBar (145 ksi tensile) on 16" centers provides 23,925 lbs of tensile restraint to the slab at the same reinforcement ratio of .46% (when considering the reduction in concrete slab thickness). This 23% increase in tensile strength combined with the reduced internal stresses (provided by its low tensile modulus) will provide improved slab performance. The benefits of this reduced tensile modulus are presented in the attached article titled: *FRP Rebar in Slabs on Grade Benefit from Low Modulus of Elasticity* by Steven E. Williams, P.E.. To summarize this paper the reduced tensile modulus of GatorBar will result in a larger crack spacing and larger crack widths. Neuvokas will typically review customer applications to verify that crack widths stay with American Association of State Highway Transportation Office (AASHTO) guidelines.

When considering crack control using a smaller diameter GatorBar with a smaller spacing will help reduce the quantity and spacing of cracks. By presenting more restraint over a greater surface area (2.2 to 1 increase) to the slab, the stress concentrations at each reinforcement spot are reduced. This results in lower chances of punch-out failure and keeps the crack width small.

GatorBar's lower tensile modulus will combat curling caused by internal stresses and because it doesn't rust, spalling from corrosion is not a concern if there are relatively more cracks.

	Rebar Size	OCBW in	Reinforcement Ratio	Quantity of Rebar ft	Tensile Restraint Offered	Total Rebar Cost
Steel rebar	#5	24	0.51%	283,140	18,420	$106,621.76
Gatorbar	#3	16	0.46%	424,710	23,925	$106,177.50
Site requires 31,460 yds2						
Reinforcement ratio based on concrete thickness and OCBW						

Table 5. Steel rebar vs. GatorBar project comparison

Economic Feasibility / Cost Analysis

Table 4 provides a summary of the concrete, base, and subbase cost for each proposal. It is assumed that all approaches use essentially the same amount of labor as the currently specified design. Proposal 1 would cost $27,437 more in subgrade cost, Proposal 2 would cost $101,054 more in subgrade cost, and Proposal 3 would cost $162,401 more in subgrade cost. Each of these proposals results in significant cost savings for the customer when considering the reduction in concrete cover that can be achieved. This cost is based on the summary of costs listed in Table 6 and 7.

	Concrete Thickness inch	Concrete Needed ft^2	Concrete	Base and Subbase	Total Cost	Total Savings
Current Specification	10	283,140	$ 1,755,468.00	$ 73,616.40	$1,829,084.40	
Proposal 1, 9% Cement	6	283,140	$ 1,189,188.00	$101,054.01	$1,290,242.01	$ 538,842.39
Proposal 2, 9% cement	6	283,140	$ 1,189,188.00	$174,670.41	$1,363,858.41	$ 465,225.99
Proposal 3, 6% cement	6	283,140	$ 1,189,188.00	$236,017.41	$1,425,205.41	$ 403,878.99
Note: Subbase cost in proposal 3 assumes importing 6" of crushed limestone.						

Table 6. Cost Summary

Concrete Cost	$ 4.20	$/ft2 6" inc. rebar
Concrete Cost	$ 2.00	$/ft2 at 4" thick
Cement Cost	$ 0.07	$/lb
Sand Cost	$18.00	$/ton
Lime Cost	$ 0.07	$/lb
Limestone cost	$30.00	$/ton

Table 7. Cost Summary of line items

Conclusion and Summary

The composite stiffness of a CTB base, whether Proposal 1 or Proposal 2, cannot match the current specification with ten inches of concrete. While this is understood, the maximum stress within the concrete in any of these proposals is still lower than the modulus of rupture so the concrete in any of them should not crack with this loading. Proposal 3 utilizing a crushed limestone CTAB can match or exceed the composite stiffness of the current specification while utilizing only six inches of concrete. Each of these proposals have been specified and used in Texas on a variety of projects, and do not represent something that has not been done before. The end result of this approach will be a slab with a high stiffness due to its six inches of "structural" base and increased concrete stiffness.

By placing the GatorBar above the centerline in the slab thickness the rebar will truly serve as crack control for Houston Texas ground thrust due to expansive clay conditions. Lastly because the subbase is more rigid the slab will benefit from increased resistance to potential pumping at expansion joints during rain events.

Viewed holistically this approach will nominally change many aspects without making a major change in any one place. The degree of difficulty to the concrete contractor installing the pavement is not significantly altered. The cost of the materials involved for any of the proposals will result in money saved. The Customer receives a superior slab that will carry the loads specified.

Remaining Concerns / Further Analysis

All calculations in this paper are for stiffness or deformation only. So they are not directly related to strength properties such as "3500 psi" or "5000 psi" for concrete strength. The stiffness only reflects the relationship between load/stress and deflection/deformation in small strain conditions. In such conditions, no plastic deformation or cracking occurs. Besides, there are many other concerns in pavement design besides deformation, such as drainage and settlement. Secondly, time and the savings available needs to be mentioned when the above evaluation is conducted. This is because the pozzolanic reactions in lime-stabilized soils are much slower than the cement hydration reactions in cement-stabilized soils allowing contractors to begin work much sooner on cement treated base. Thirdly, changes in material properties may change the above results considerably.

Another potential issue is differential settlement. This could be caused by either the variability of soil properties across the region or by a long-term load distributed over a specific area. Assuming the FAT clay has a medium to high compressibility, that is, a compressibility ratio of 0.2. The 10 feet clay layer could generate a consolidation settlement from several inches to a couple of feet, which, over the time scale of several months to

several decades, could manifest itself, depending on the level of loading, the coefficient of consolidation of the clay, and local drainage condition. If a pile of steel is placed over an area for a long time, the underlying clay layer could produce significant consolidation settlement while that below other unloaded areas produce much less or even negligible settlement. This differential settlement can bend the concrete slab. Depending on the magnitude of the differential settlement, cracks could be initiated and propagated along the margins between the loading and unloading areas.

Supporting Data

In its preliminary research Neuvokas uncovered several articles concerning projects that used these approaches to great effect. In particular the San Antonio Texas spur 66/Watson Road project and the West Virginia Route 9 project outside Martinsville have well documented the use of CTB. Various sources also list different cement treatment levels and methods to place the Portland cement. Document TM5-822-14 from the Army and Air Force further explain this method for soil stabilization.   TxDOT has published specification 275 that explains the placement of CTB as well.

References

Little, Dallas. *Evaluation of Structural Properties of Lime Stabilized Soils*, National Lime Association, Jan. 1999.

Muthunthan, Balasingam and Farid Sarioesseiri, *Interpretation of Geotechnical Properties of Cement Treated Soils*, Washington State Department of Transportation, July 2008.

Lim, Seungwook and Dan Zollniger, *Estimation of the Compressive Strength and Modulus of Elasticity of Cement-Treated Aggregate Base Materials*, TRB 2003 Annual Meeting, 2003.

Bhattacharja, Sankar and Javed Bhatty, *Comparative Performance of Portland Cement and Lime Stabilization of Moderate to High Plasticity, Portland Cement Association*, 2003.

AASHTO, *AASHTO Guide for Design of Pavement Structures*, American Association of State Highway and Transportation Officials, Washington, D.C.

ACPA, *Stabilized Subbases*, Concrete Pavement Technology Series,. 2008

Army, TM 5-809-1, *Concrete Floor Slabs on Grade Subjected to Heavy Loads*, , Chapter 5 page 5-1 to 5-3, August 1987.

Little, Dallas, *Recommended Practice for Stabilization of Subgrade Soils and Base Materials*. NCHRP, August 2009.

http://www.cement.org/think-harder-concrete-/paving/cement-modified-soils-(cms)/cms-case-histories/cement-speeds-up-stabilizes-txdot-san-antonio-project

Soil Mechanics Equations
Constitutive Relationship: Stress-Strain Relationship

The deflection of the pavement structure and underlying subgrade is a very typical multi-region linear elastic problem. For this elastic material, the general constitutive relationship is Hooke's law:

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon},$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{\sigma}$ is the infinitesimal strain tensor, and $\mathbf{C}$ is the fourth-order stiffness tensor. However, in engineering applications, we usually do not deal with the above equation using the stress and strain as second order tensors and the stiffness as the forth order tensors.

Here we assume both concrete and soils including treated one are homogeneous and isotropic materials, then the stiffness tensor can be written using the Vigot notation as:

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\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}
=
\begin{bmatrix}
2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\
\lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu
\end{bmatrix}
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}
$$

where μ and λ are Lamé constants. These two constants can be formulated as functions of Young's modulus and Poisson's ratio, which we are more familiar with.
In index notation, we can write the above equation as

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}.$$

The Governing Equation: Navier's Equation

The governing equation for the mechanical field can be obtained based on the balance law: Cauchy's equations of motion. To be more specific, according to the principle of conservation of linear momentum, if the continuum body is in static equilibrium it can be demonstrated that the components of the Cauchy stress tensor in every material point in the body satisfy the equilibrium equations. Let us consider a continuum body occupying a volume V, having a surface area S, with defined traction or surface forces $T_i^{(n)}$ per unit area acting on every point of the body surface, and body forces F_i per unit of volume on every point within the volume V. Thus, if the body is at equilibrium the resultant force acting on the volume is zero, thus:

$$\int_S T_i^{(n)} dS + \int_V F_i dV = 0$$

By definition the stress vector is $T_j^{(n)} = \sigma_{ji} n_j$, then

$$\int_S \sigma_{ji} n_j dS + \int_V F_i dV = 0$$

Using the Gauss's divergence theorem to convert a surface integral to a volume integral gives

$$\int_V \sigma_{ji,j} dV + \int_V F_i dV = 0$$

$$\int_V \left(\sigma_{ji,j} + F_i \right) dV = 0$$

For an arbitrary volume the integral vanishes, and we have the equilibrium equations:

$$\sigma_{ji,j} + F_i = 0 \, .$$

But as mentioned above, we usually deal with displacement directly, which requires the displacement formation of Navier's equation.

In this case, the displacements are prescribed everywhere in the boundary. In this approach, the strains and stresses are eliminated from the formulation, leaving the displacements as the unknowns to be solved for in the governing equations. First, the strain-displacement equations are substituted into the constitutive equations (Hooke's Law), eliminating the strains as unknowns:

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} = \lambda \delta_{ij} \varepsilon_{k,k} + \mu \left(u_{i,j} + u_{j,i} \right).$$

Differentiating yields:

$$\sigma_{ij,j} = \lambda \varepsilon_{k,ki} + \mu \left(u_{i,jj} + u_{j,ij} \right).$$

Substituting into the equilibrium equation yields:

$$\lambda u_{k,ki} + \mu \left(u_{i,jj} + u_{j,ij} \right) + F_i = 0$$

or

$$\mu \varepsilon_{k,ki} + \left(\mu + \lambda \right) \left(u_{i,jj} + u_{j,ij} \right) + F_i = 0$$

The governing is also called Navier-Cauchy equations or the elastostatic equations. The above equation is also frequently formulated using the tensor notation as

$$\mu \nabla^2 \mathbf{u} + \left(\mu + \lambda \right) \nabla \left(\nabla \cdot \mathbf{u} \right) + \mathbf{F} = 0.$$

This is the primary equation using for the simulation.